

# Apsidal Alignment in Upsilon Andromedae

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## ABSTRACT

One of the parameters fitted by Doppler radial velocity measurements of extrasolar planetary systems is  $\omega$ , the argument of pericenter of a given planet's orbit referenced to the plane of the sky. Curiously, the  $\omega$ 's of the outer two planets orbiting Upsilon Andromedae are presently nearly identical:  $\Delta\omega \equiv \omega_D - \omega_C = 4.8 \pm 4.8(1\sigma)$ . This observation is least surprising if planets C and D occupy orbits that are seen close to edge-on ( $\sin i_C, \sin i_D \gtrsim 0.5$ ) and whose mutual inclination  $\Theta$  does not exceed  $20^\circ$ . In this case, planets C and D inhabit a secular resonance in which  $\Delta\omega$  librates about  $0^\circ$  with an amplitude of  $\sim 30^\circ$  and a period of  $\sim 4 \times 10^3$  yr. The resonant configuration spends about one-third of its time with  $|\Delta\omega| \leq 10^\circ$ . If  $\Theta \gtrsim 40^\circ$ , either  $\Delta\omega$  circulates or the system is unstable. This instability is driven by the Kozai mechanism which couples the eccentricity of planet C to  $\Theta$  to drive the former quantity to values approaching unity. Our expectation that  $\Theta \lesssim 20^\circ$  suggests that planets C and D formed in a flattened, circumstellar disk, and may be tested by upcoming astrometric measurements with the FAME satellite.

*Subject headings:* celestial mechanics — planetary systems — stars: individual (*v* Andromedae)

## 1. INTRODUCTION

Upsilon Andromedae (*v* And) is a Sun-like star harboring at least three planetary companions (Butler et al. 1999). The star's radial velocity variations are fitted approximately by the superposition of three Keplerian sinusoids, each of which takes the form

$$V = K[\cos(f + \omega) + e \cos \omega]. \quad (1)$$

Here

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$$K = \frac{m \sin i}{M_* + m} \sqrt{\frac{G(M_* + m)}{a(1 - e^2)}} , \quad (2)$$

$f$  is the true anomaly of a given planet,  $i$  is the unknown inclination between this planet’s orbit plane and the plane of the sky, and the quantities  $G$ ,  $M_* = 1.3M_\odot$ ,  $m$ ,  $a$ , and  $e$  take their usual meanings. Definitions and current fitted values of orbital parameters are listed in Table 1, as kindly supplied by G. Marcy and D. Fischer (2001).

The quantity  $\omega$  in equation (1) and Table 1 is a given planet’s argument of pericenter referenced to the plane of the sky:

$$\omega = \text{sign}[(\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \cdot \hat{\mathbf{l}}] \arccos(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}) \in (-\pi, \pi] \quad (3)$$

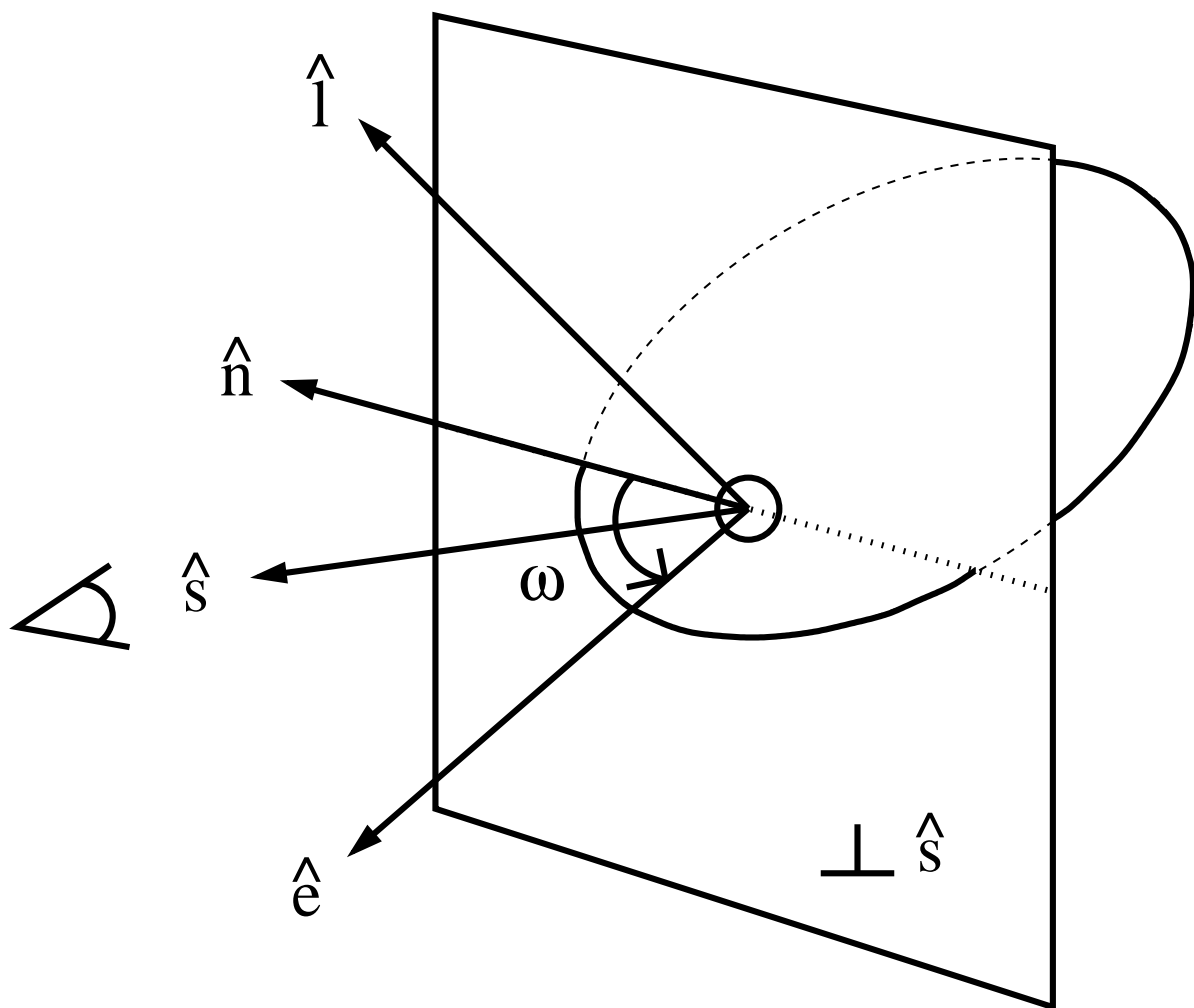
where

$$\hat{\mathbf{n}} \equiv \frac{\hat{\mathbf{s}} \times \hat{\mathbf{l}}}{\sqrt{1 - (\hat{\mathbf{s}} \cdot \hat{\mathbf{l}})^2}} . \quad (4)$$

Let the origin be positioned at the barycenter of  $v$  And and take the sky plane to contain this origin. Then  $\hat{\mathbf{s}}$  is the unit vector pointing from the origin to the observer;  $\hat{\mathbf{l}}$  is the unit vector parallel to the planet’s orbital angular momentum;  $\hat{\mathbf{n}}$  is the unit vector directed from the origin to the node of the orbit on the sky plane at which the planet is approaching the observer; and  $\hat{\mathbf{e}}$  is the unit vector pointing from the origin to the pericenter of the planet’s orbit. Figure 1 illustrates the geometry. A single value of  $\omega$  corresponds to a volume of allowed parameter space swept out by vectors  $\hat{\mathbf{s}}$ ,  $\hat{\mathbf{l}}$ , and  $\hat{\mathbf{e}}$ .

Curiously, the  $\omega$ ’s of the outer two planets C and D are presently nearly identical:  $\Delta\omega = \omega_D - \omega_C = 4.8 \pm 4.8 (1\sigma)$ . Let us define  $\Theta = \arccos(\hat{\mathbf{l}}_C \cdot \hat{\mathbf{l}}_D)$  to be the mutual inclination between the two orbit planes of planets C and D. If we assume for the moment that  $\Theta = 0^\circ$ , so that the orbits are co-planar and “co-rotating,” then the observation that  $|\Delta\omega| \ll 1$  rad implies the near perfect alignment of orbital pericenters. We may ask whether the alignment is merely coincidental, or whether a dynamical mechanism exists to lock the apsidal lines together.

Precedents for apsidal locking exist in the Solar System. Eccentric planetary rings precess rigidly about Uranus and Saturn; the inner and outer elliptical edges of a given ring maintain the same line of apsides by a balance of forces due to the quadrupole field of the central planet, ring self-gravity, and interparticle collisions (Chiang & Goldreich 2000). Of greater relevance to the case of  $v$  And is the observation that the periapsides of some asteroids librate (undergo small oscillations) about the apsidal line of Jupiter (Milani & Nobili 1984). Such asteroids inhabit a secular apsidal resonance in which their time-averaged apsidal precession rates match that of Jupiter.



Planets C and D of  $\nu$  And might occupy a similar apsidal resonance. How might we use this possibility to constrain the origin of these planets? Suppose that such a resonance operates if  $\Theta = 0^\circ$ , and suppose that it does not operate otherwise. In the latter case, suppose that  $\Delta\omega$  circulates (cyclically runs the gamut between  $0^\circ$  and  $360^\circ$ ), so that the probability of finding  $|\Delta\omega| \ll 1$  rad at any given time is small. Given these assumptions, the observation today that  $|\Delta\omega| \ll 1$  rad might lead us to suspect that  $\Theta = 0^\circ$ . Co-planarity would argue, in turn, that planets C and D formed from a flattened, circumstellar disk.

Even if the resonant dynamics are fully understood, this line of reasoning is suggestive at best. A quantitative, Bayesian assessment of the *a posteriori* probability distribution for  $\Theta$  given the observation that  $|\Delta\omega| \ll 1$  rad requires knowledge of the *a priori* probability distribution for  $\Theta$ . The latter distribution depends upon the unknown conditions of the formation of these two planets—the very object of our inferences.

This paper examines the long-term evolution of the orbits of planets D, C, and to a limited extent B, within the multi-dimensional parameter space of present-day viewing and orbital geometries that are permitted by the Doppler observations. We seek to identify regions in this parameter space—in particular, all ranges of the mutual inclination  $\Theta$ —that correspond to orbital evolutions for which  $|\Delta\omega| \ll 1$  rad for all time; we suggest that the planets of  $\nu$  And reside in regions that are characterized by such apsidal resonances. The volume of parameter space to be surveyed is substantial; for instance, if  $\Theta \neq 0^\circ$ , then  $\omega_D = \omega_C$  does not even necessarily imply that  $\hat{\mathbf{e}}_C$ ,  $\hat{\mathbf{e}}_D$ ,  $\hat{\mathbf{l}}_C$ , and  $\hat{\mathbf{l}}_D$  lie in the same plane. Equality of  $\omega$ 's merely reflects the equality of angles which may well be referenced to different nodes and which may therefore have no direct physical relation.

Previous explorations of this parameter space by Rivera & Lissauer (2000ab) and Stepinski, Malhotra, & Black (2000) have indeed uncovered the existence of a secular apsidal resonance between planets C and D in the  $\Theta = 0^\circ$  case. Furthermore, considerations of dynamical stability presented by these authors limit values of  $\sin i$  in the  $\Theta = 0^\circ$  case to be greater than  $\sim 0.3$ . Stability considerations also argue against  $\Theta \gtrsim 60^\circ$  (Stepinski et al. 2000).

Our analysis extends and improves upon these previous results in a number of respects. First, we take explicit account of the dependence of  $\omega$  on viewing geometry. To our knowledge, the only study to mention this dependence is that of Rivera & Lissauer (2000a), who incorporate it in their calculations of co-planar, counter-rotating orbits ( $\Theta = 180^\circ$ ). Second, we systematically explore all of the parameter space spanned by possible orbital configurations of planets C and D and by possible lines-of-sight; previous studies explore only a fraction of this space. Third, we focus on the evolution of  $\Delta\omega$  as a means of distinguishing between likely and unlikely orbital configurations. As we shall see, the existence of a secular apsidal resonance characterized by small libration amplitude in select regions of parameter space suggests tighter constraints on  $\Theta$  than considerations of stability alone. Finally, the Doppler-fitted values of orbital parameters that we employ are improved over those used in previous studies due to the increased number of data points ( $N = 189$  radial velocity measurements as of January 2001; Marcy & Fischer 2001).

In §2, we take  $\Theta$  to be zero and study the secular evolution of the orbits of planets C and D while neglecting the presence of the innermost planet B. We recover analytically the secular resonance found numerically by Rivera & Lissauer (2000ab) and Stepinski et al. (2000) and describe its physical character. In §3, we allow  $\Theta \neq 0^\circ$  while continuing to ignore planet B. All viewing/orbital configurations for which  $\Delta\omega$  potentially librates about  $0^\circ$  are identified. In §4, we ask which of the resonant configurations found in §3 are likely to remain resonant with the inclusion of planet B. Section 5 marshals our results to argue that the planets of  $\nu$  And most likely originated in a circumstellar disk.

## 2. Planets C and D: $\Theta = 0^\circ$

In the co-planar, co-rotating case, the individual  $\omega$ 's represent longitudes of pericenters referenced to a common nodal vector  $\hat{\mathbf{n}}$  and a common orbital plane. We ask in this case whether  $\Delta\omega$  circulates or librates, and investigate how the answer depends on uncertainties in the orbital parameters, including  $\sin i$ .

### 2.1. Analytic Description

Long-term variations in the eccentricities and apsidal longitudes of the outer two planets are qualitatively well described by the classical secular solution of Laplace-Lagrange (L-L; see, e.g., Murray & Dermott 1999). Let  $q = m_D/m_C = 2.03 \pm 0.05$  ( $1\sigma$ ),  $\alpha = a_C/a_D = 0.3238 \pm 0.0009$  ( $1\sigma$ ), and let  $\beta = b_{3/2}^{(2)}(\alpha)/b_{3/2}^{(1)}(\alpha) = 0.476/1.19 = 0.399 \pm 0.001$  ( $1\sigma$ ) be the ratio of Laplace coefficients (Brouwer & Clemence 1961). These quantities are evaluated using the orbital parameters of Marcy & Fischer (2001). Then

$$e_C \exp i\omega_C = e_{C+} \exp i(g_+t + \gamma_+) + e_{C-} \exp i(g_-t + \gamma_-) \quad (5)$$

$$e_D \exp i\omega_D = e_{D+} \exp i(g_+t + \gamma_+) + e_{D-} \exp i(g_-t + \gamma_-) \quad (6)$$

where the frequencies  $g_+$  and  $g_-$ , and ratios  $e_{D+}/e_{C+}$  and  $e_{D-}/e_{C-}$ , are constants specified by the masses and secularly invariant semi-major axes of the two planets:

$$g_{\pm} = \frac{\pi m_C}{4M_* P_C} \alpha^2 b_{3/2}^{(1)}(\alpha) \left\{ q + \sqrt{\alpha} \pm \sqrt{(q - \sqrt{\alpha})^2 + 4q\sqrt{\alpha}\beta^2} \right\} \approx \begin{cases} 2\pi/(7000 \sin i \text{ yr}) \\ 2\pi/(30000 \sin i \text{ yr}) \end{cases} \quad (7)$$

$$\left( \frac{e_D}{e_C} \right)_{\pm} = \frac{q - \sqrt{\alpha} \mp \sqrt{(q - \sqrt{\alpha})^2 + 4q\sqrt{\alpha}\beta^2}}{2q\beta} \approx \begin{cases} -0.14 \\ 1.9 \end{cases} . \quad (8)$$

Here  $P_C$  is the orbital period of planet C. These equations neglect terms of order  $e^3$  and  $(m/M_*)^2$ . The Laplace-Lagrange description is valid for small values of the mutual inclination  $\Theta$  provided  $\omega$  is replaced by  $\tilde{\omega}$ , the longitude of pericenter referenced to the invariable plane (the usual “dog-leg” angle defined in, e.g., Murray & Dermott 1999).

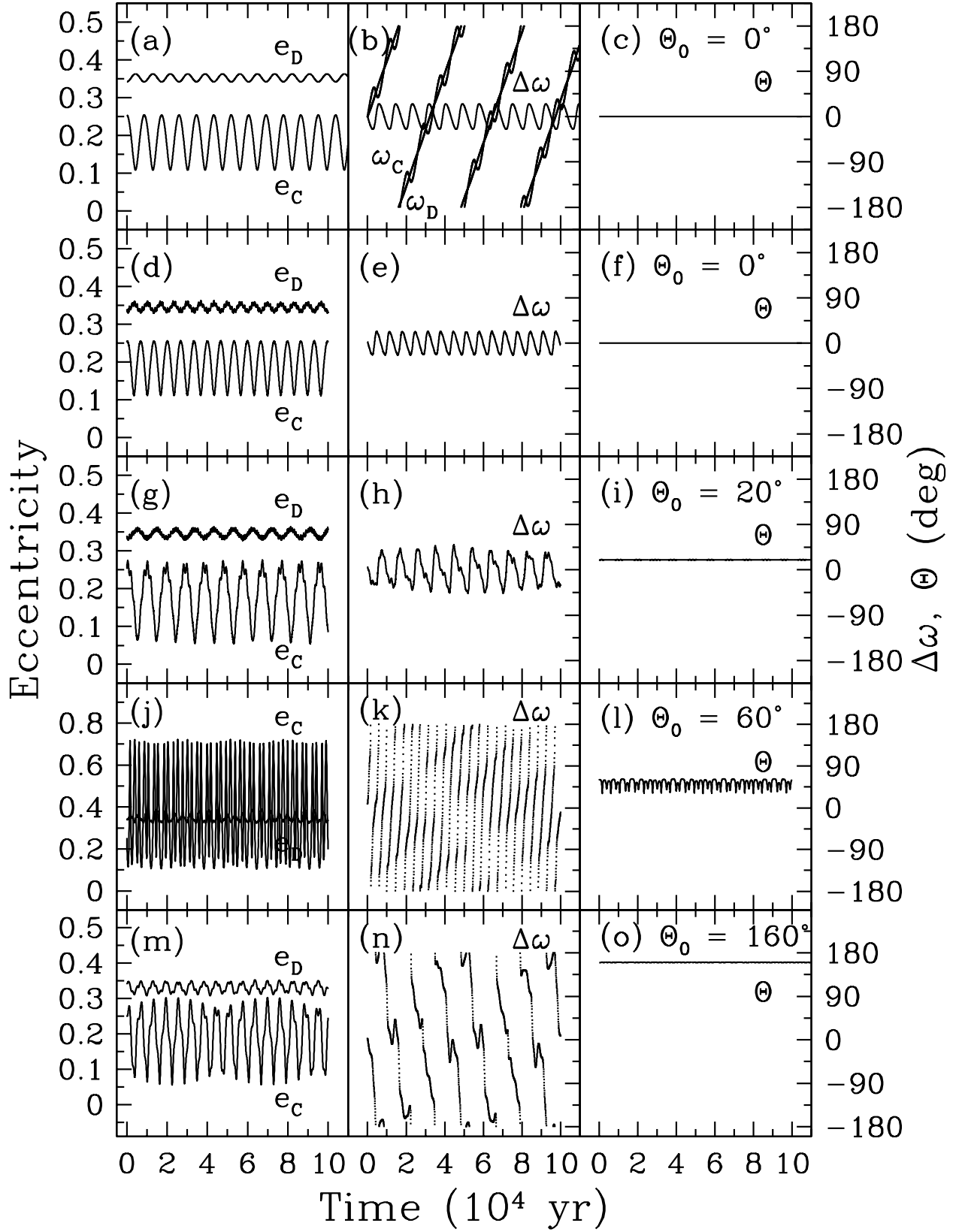
In writing equations (5)–(8), we have omitted terms due to the third, innermost planet B. Provided planet B remains on an orbit that is more than a factor of 10 smaller than the orbits of the outer two bodies, its time-averaged potential acts mainly as a static quadrupole moment. This quadrupole could add corrections up to order  $(m_B/m_C)(a_B/a_C)^2 \sim$  a few percent to our expressions for the eigenfrequencies (7) and eigenvector amplitude ratios (8). In this section and §3, we neglect the effects of planet B for clarity of presentation and computational ease. We incorporate quantitatively the effects of planet B in §4.

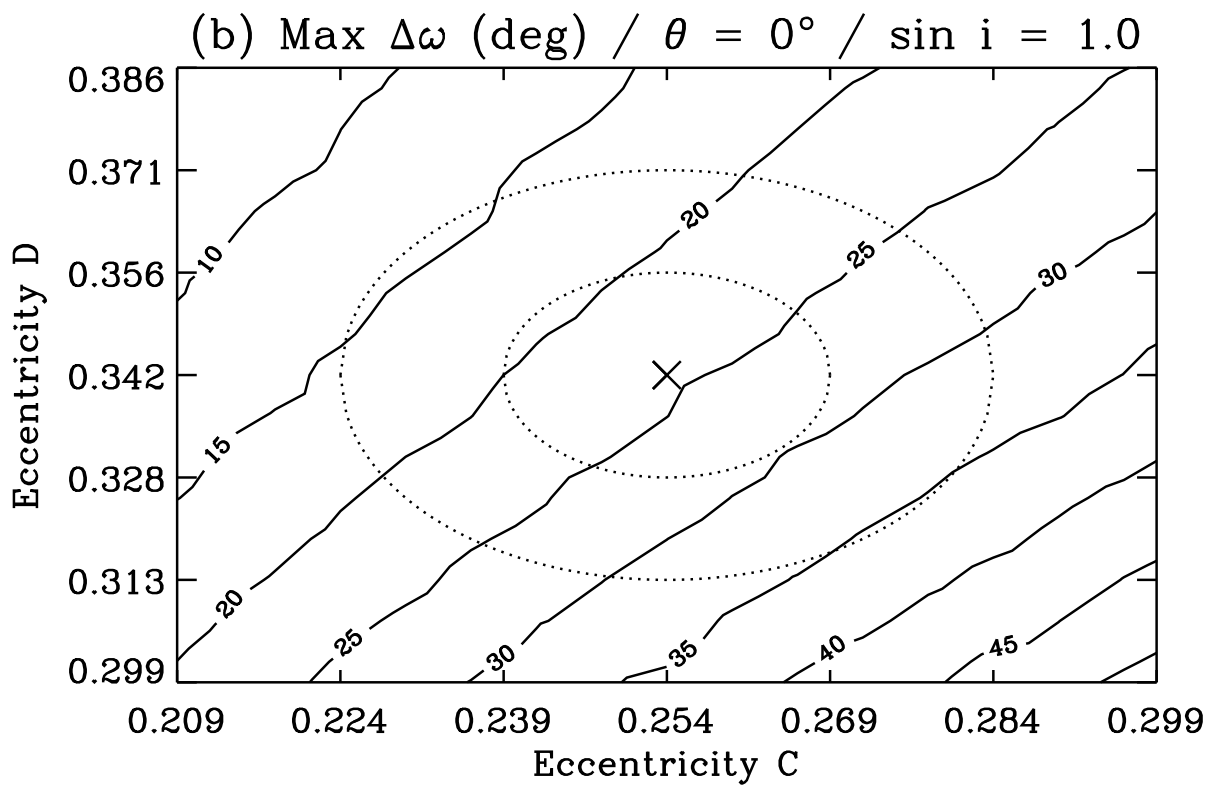
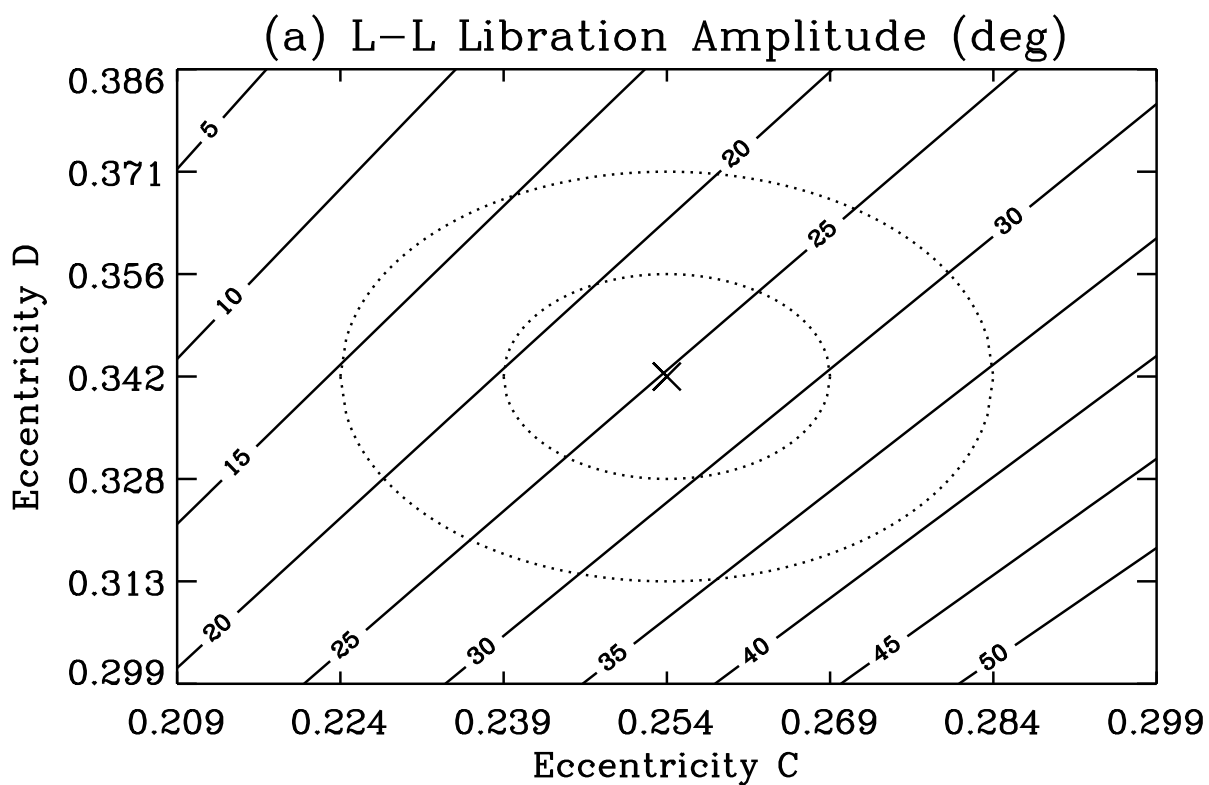
The remaining four constants  $\gamma_{\pm}$ ,  $e_{C+}/e_{C-}$ , and  $e_{D+}/e_{D-}$  require specification of the initial eccentricities and apsidal longitudes. If initial conditions are such that  $|e_{C+}/e_{C-}| \ll 1$  and  $|e_{D+}/e_{D-}| \ll 1$ , then equations (5)–(6) imply that the two planets precess in lockstep at frequency  $g_-$  with fixed eccentricities and aligned pericenters. The present-day orbital parameters of  $v$  And are such that most, but not all, of the power is in the “–” eigenmode. Setting  $\omega_C = 0^\circ$ ,  $\omega_D = 4^\circ 8$ ,  $e_C = 0.25$ , and  $e_D = 0.34$  at  $t = 0$ , we obtain with a numerical root-finder:

$$\gamma_+ = -10^\circ, \quad \gamma_- = 4^\circ 3, \quad e_{D+}/e_{D-} = -0.030, \quad e_{C+}/e_{C-} = 0.41. \quad (9)$$

The evolution described by equations (5)–(9) is plotted in Figures 2abc. On average, both  $\omega_D$  and  $\omega_C$  advance at the same rate  $g_-$ . In addition, however, the admixture of power from the “+” mode ( $e_{C+}/e_{C-}$  is not substantially smaller than unity) implies that  $\omega_C$  librates about  $\omega_D$  at a faster frequency  $\sim g_+$ . The argument of pericenter of planet D librates less since that planet carries the bulk ( $\sim 75\%$ ) of the angular momentum in the system.

The apsidal libration amplitude in L-L theory is a function of the initial individual eccentricities ( $e_{C0}$  and  $e_{D0}$ ), the initial difference in apsidal longitudes ( $\Delta\omega_0$ ),  $\alpha$ , and  $q$ . For planets C and D, most of the uncertainty in the L-L libration amplitude arises from uncertainties in the fitted eccentricities. Figure 3a exhibits the variation of L-L libration amplitude with  $e_{C0}$  and  $e_{D0}$ , for fixed  $\alpha = 0.324$ ,  $K_D/K_C = 1.19$ , and  $\Delta\omega_0 = 0^\circ$ . Allowance is made for the fact that for fixed  $\alpha$  and  $K_D/K_C$ , the observed  $q = m_D/m_C \propto \sqrt{1 - e_{D0}^2} / \sqrt{1 - e_{C0}^2}$  according to equation (2). We take  $\Delta\omega_0$  to be  $0^\circ$  because this approximation permits the immediate solution of equations (5)–(8). Variations in libration amplitude with  $\Delta\omega_0 \in [-4^\circ 8, 14^\circ]$  ( $2\sigma$  confidence interval) are typically less than 10% (other parameters being held fixed); these variations are small compared to the  $\sim 50\%$  variation in libration amplitude associated with the  $2\sigma$  confidence region in fitted eccentricities. Variations in libration amplitude associated with uncertainties in  $\alpha$  and  $K_D/K_C$  are negligibly small, at the level of  $\sim 0.3\%$  and  $\sim 3\%$ , respectively. Since the L-L libration amplitude is sensitive to the mass ratio but not to the individual masses, it is independent of  $\sin i$  in the co-planar case.







The L-L amplitude associated with the best-fit eccentricities is  $25^\circ$ . The corresponding fraction of time that the system spends with  $|\Delta\omega| \leq 10^\circ$  equals  $P_{time} = 0.26$ . Within the  $2\sigma$  error ellipse of fitted eccentricities, the amplitude ranges from  $14^\circ$  ( $P_{time} = 0.52$ ) to  $38^\circ$  ( $P_{time} = 0.16$ ).

We conclude from this analytic study that if planets C and D occupy co-planar, co-rotating orbits, then they inhabit a secular resonance for which  $\Delta\omega$  librates about  $0^\circ$  with an amplitude of  $25^\circ \pm 6^\circ$  ( $1\sigma$ ). In other words, if  $\Theta = 0^\circ$ , the observation today that  $|\Delta\omega| \leq 10^\circ$  should not be regarded as surprising, since the best-fit system spends about one-quarter of its time in this state. These libration parameters change slightly as we incorporate more realistic effects in subsequent sections (e.g., the presence of planet B is included in §4).

## 2.2. Numerical Solution

We verify the analytical considerations above by numerically integrating the orbits of planets C and D using the SWIFT software package developed by Levison & Duncan (1994). We employ their mixed variable symplectic integrator, which is derived from the algorithm invented by Wisdom & Holman (1991). As before, the effects of planet B are neglected. The timestep is sufficiently small ( $\Delta t = 3$  days) that the total energy in the system is conserved to within 1 part in  $10^7$  after  $t = 10^6$  yr. Initial conditions are taken from the best-fit parameter values in Table 1. The starting epoch of our integrations was chosen arbitrarily to be JD 2450160.1 days. Since the effects in which we are primarily interested are secular (orbit-averaged), our results should be qualitatively insensitive to choice of starting epoch. We have verified that none of our conclusions changes by choosing an alternative starting epoch of JD 2451160.1 days. Moreover, since substantial changes in the osculating orbital elements of planets C and D occur only over secular timescales that are long compared to the duration of the radial velocity observations, the errors in the fitted orbital parameters in Table 1 introduced by the assumption of fixed Keplerian ellipses are no more than several percent (Laughlin 2001; see also Laughlin & Chambers 2001).

As seen in Figures 2def, the behavior of  $\Delta\omega$  with time for  $\sin i = 1$  computed using SWIFT is similar to that predicted by L-L. For smaller values of  $\sin i \in [1, 0.3]$  and correspondingly greater planetary masses, the system remains in the secular apsidal resonance, but short period terms in the disturbing potentials effectively increase the amplitude of libration. Thus, the fraction of time for which  $|\Delta\omega| \leq 10^\circ$  decreases from  $P_{time} = 0.30$  to 0.20 as  $\sin i$  decreases from 1 to 0.3. For  $\sin i \lesssim 0.3$ , we find, as do Rivera & Lissauer (2000b) and Stepinski et al. (2000), that the system is unstable.

Figure 3b plots the maximum value of  $\Delta\omega$  over  $10^5$  yr as a function of the initial eccentricities for  $\sin i = 1$ . The contours are similar to those predicted by L-L, with deviations introduced by short period terms and higher order secular terms. The range in  $P_{time}$  within the  $2\sigma$  error ellipse of fitted eccentricities is  $[0.20, 0.54]$ .

### 3. Planets C and D: $\Theta \neq 0^\circ$

How does the secular evolution of  $\Delta\omega$  change with mutual orbital inclination,  $\Theta$ ? In particular, for what ranges of  $\Theta$  can planets C and D occupy an apsidal resonance? We answer these questions using Monte-Carlo sampling techniques and numerical orbit integrations. As in §2, we neglect the effects of planet B. We begin by delineating the parameter space spanned by viewing geometries and orbital configurations of planets C and D that are allowed by the Doppler observations. Subsequent subsections describe (1) the method by which we sample the allowed volume of parameter space and (2) the dynamics that play out within this volume.

#### 3.1. Geometrical Considerations

Let the set of “initial orientations” refer to the set of  $\{\hat{\mathbf{l}}_C, \hat{\mathbf{l}}_D, \hat{\mathbf{e}}_C, \hat{\mathbf{e}}_D, \hat{\mathbf{s}}\}$  such that  $\hat{\mathbf{e}}_D \cdot \hat{\mathbf{l}}_D = \hat{\mathbf{e}}_C \cdot \hat{\mathbf{l}}_C = 0$  and  $\hat{\mathbf{l}}_D \cdot \hat{\mathbf{l}}_C = \cos \Theta_0$ . The subscript “0” denotes the initial or present-day value; as we shall see, the mutual inclination varies with time in most circumstances. These orbital and viewing orientation vectors sweep out a volume in the following five dimensions. Without loss of generality, we erect a set of Cartesian coordinate axes  $\hat{\mathbf{x}} = \hat{\mathbf{e}}_C$ ,  $\hat{\mathbf{z}} = \hat{\mathbf{l}}_C$ , and  $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ . Then the five dimensions of initial orientation space are  $\Theta_0, \Omega_0, \chi_0$ —the initial inclination, initial longitude of ascending node, and initial argument of pericenter, respectively, of the orbit of planet D upon that of planet C—and  $\phi, \delta$ —the azimuthal and polar angles of  $\hat{\mathbf{s}}$ . In the absence of observational constraints, the angles  $\Omega_0, \chi_0, \phi \in [0, 2\pi)$ , while  $\Theta_0, \delta \in [0, \pi]$ .

The observed present-day values of  $\omega_C$  and  $\omega_D$  carve out a subset in initial orientation space which we call the set of “allowed initial orientations.” We combine this set with the best-fit values of the remaining measured orbital parameters of planets C and D to define the set of “allowed initial conditions.” Each member of the set of allowed initial conditions may be used as initial conditions for the SWIFT orbit integrator. We refer to the resultant evolution as a “scenario.”

#### 3.2. Sampling

For each of 19 values of  $\Theta_0$  spaced uniformly between  $0^\circ$  and  $180^\circ$ , we randomly generate sets of  $(\Omega_0, \chi_0, \phi, \delta)$  using the following normalized probability distribution functions. Isotropy of physical space demands that  $\partial P / \partial \phi = 1/2\pi$ ,  $\partial P / \partial (\cos \delta) = 1/2$ , and  $\partial P / \partial \Omega_0 = 1/2\pi$ ; i.e.,  $\phi$ ,  $\cos \delta$ , and  $\Omega_0$  are uniformly distributed over their respective domains. The probability distribution function for  $\chi_0$ , like that for  $\Theta_0$ , is unknown (and may well depend on other parameters such as  $\Omega_0$ ), reflecting the unknown physics of the formation of these planets. We assume for simplicity that  $\chi_0$  is uniformly distributed over its domain; i.e.,  $\partial P / \partial \chi_0 = 1/2\pi$ .

For a given  $\Theta_0$ , those randomly generated values of  $(\Omega_0, \chi_0, \phi, \delta)$  which satisfy the observed present-day values of  $\omega_C \in [-111^\circ.1, -106^\circ.9] (1\sigma)$  and  $\omega_D \in [-106^\circ.9, -101^\circ.1] (1\sigma)$  [as computed

using equation (3)] constitute our sample of allowed initial orientations. For our choice of  $\partial P/\partial \chi_0$ , we find empirically that the fraction of initial orientations that are allowed does not vary with  $\Theta_0$ . At every  $\Theta_0$ , the first  $N = 45$ , randomly generated, allowed initial conditions are employed as input conditions for the SWIFT orbit integrator. We believe that our choice for  $N$  adequately samples parameter space since we have taken  $N = 135$  at several values of  $\Theta_0$  and find none of our conclusions below to be changed. As discussed in §2.2, the starting epoch of all integrations was JD 2450160.1 days.

### 3.3. Dynamics

Each orbit integration lasts  $t = 10^6$  yr. In addition to recording the usual orbital parameters such as the eccentricities and mutual inclination, we also compute  $\Delta\omega$  according to equation (3). The integration spans  $10^2$ – $10^3$  libration/circulation periods of  $\Delta\omega$ ; instabilities usually manifest themselves on shorter timescales. A system is considered to be unstable if either planet collides with the central star or if either planet is ejected from the system. In practice, we take the latter criterion to be satisfied if the semi-major axis of the orbit exceeds 20 AU; we have checked *a posteriori* that stable scenarios preserve semi-major axes to within  $\sim 10\%$ . Integrating  $45 \times 19 = 855$  scenarios, each for  $t = 10^6$  yr with a timestep of  $\Delta t = 3$  days, requires 6 CPU days on a DEC Alpha workstation.

Secular libration of  $\Delta\omega$  about  $0^\circ$  is a typical outcome for  $\Theta_0 \lesssim 30^\circ$ . Figures 2ghi display the evolution of eccentricities,  $\Delta\omega$ , and  $\Theta$  in one example scenario for which  $\Theta_0 = 20^\circ$ . The caption to Figure 2 lists the employed values of  $\sin i_C$  and  $\sin i_D$ . The gross features of the evolution are the same as those for the co-planar case (cf. Figures 2def). Because the Doppler-measured  $\omega$  is sensitive to both  $\hat{\mathbf{e}}$  and  $\hat{\mathbf{l}}$ ,  $\Delta\omega$  is modulated not just at the single usual frequency  $\sim g_+$ , but also at the two remaining secular precession frequencies,  $\sim g_-$  and the nodal precession frequency

$$\dot{\Omega} \approx -\frac{\pi m_C}{2M_* P_C} \alpha^2 b_{3/2}^{(1)}(\alpha)(q + \sqrt{a}) \approx -2\pi/(6000 \text{ yr}) . \quad (10)$$

The above expression is derived to lowest order in  $\Theta$  and the numerical evaluation takes  $\sin i_C \approx \sin i_D \approx 1$  (as is appropriate for the scenario in Figures 2ghi). In Figure 2i, the mutual inclination  $\Theta$  remains constant at  $20^\circ$ ; secular invariance of  $\Theta$  obtains for 2-planet systems with small  $\Theta$  and small eccentricities (Brouwer & Clemence 1961).

Circulation of  $\Delta\omega$ , instability, and, more rarely, libration of  $\Delta\omega$  are three possible outcomes for  $40^\circ \lesssim \Theta_0 \lesssim 90^\circ$ . The qualitatively distinct character of the disturbing function at such large inclinations was first explored by Kozai (1962), who described the secular evolution of highly inclined orbits of asteroids in Jupiter’s gravitational field. Here planet D, the main repository of angular momentum, behaves as Jupiter, and planet C behaves as an asteroid. For mutual inclinations such that  $\cos^2 \Theta \lesssim 3/5$  ( $\approx \cos^2 39.2^\circ$ ), the system may inhabit the Kozai resonance, for

which  $\hat{\mathbf{l}}_C$  and  $\hat{\mathbf{e}}_C$  precess about the axis parallel to the total angular momentum vector of the system at a common frequency  $\sim|\dot{\Omega}|$ , while  $\hat{\mathbf{e}}_D$  precesses about that axis at the slower frequency  $\sim g_-$ . The result is that  $\Delta\omega$  circulates at  $\sim|\dot{\Omega}|$ . Figures 2jkl show how the eccentricities,  $\Delta\omega$ , and  $\Theta$  evolve under the Kozai mechanism. Note how  $e_C$  and  $\Theta$  undergo larger variations than in the small  $\Theta$  case;  $e_C$  and  $\Theta$  can be coupled in the  $\cos^2\Theta \lesssim 3/5$  case such that both vary secularly at frequency  $\sim 2|\dot{\Omega}|$  while keeping Kozai’s integral,  $\sqrt{1 - e_C^2} \cos\Theta$ , approximately constant. The secular driving of  $e_C$  to values approaching unity causes 90% of our sampled scenarios having  $\Theta_0 \approx 90^\circ$  to self-destruct within  $t = 10^6$  yr.

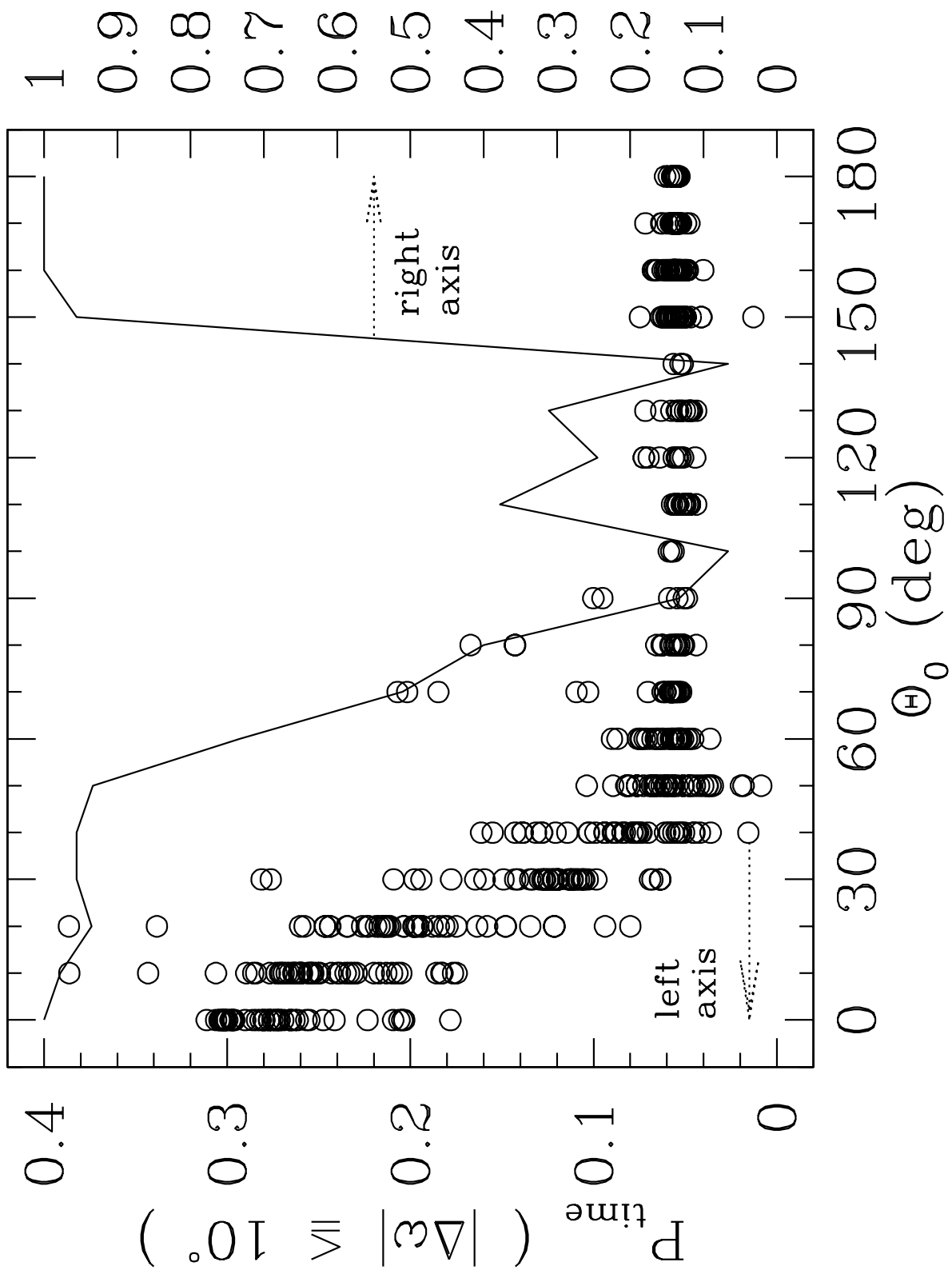
For  $\Theta_0 \gtrsim 90^\circ$ , the angular momenta of planets C and D, projected onto the total angular momentum vector of the system, point in opposite directions. Consequently, secular exchanges of angular momenta between the two planets cause their eccentricities to rise and fall together (Rivera & Lissauer 2000a). For  $90^\circ \lesssim \Theta_0 \lesssim 140^\circ$ , the synchronized oscillations of the eccentricities compounds the destabilizing effects of coupled  $e_C$  and  $\Theta$  to render the majority of scenarios unstable. For  $150^\circ \lesssim \Theta_0 \lesssim 180^\circ$ ,  $e_C$  and  $\Theta$  decouple and  $\Delta\omega$  circulates stably. A typical example of the evolution in the nearly co-planar, counter-rotating case is showcased in Figures 2mno.

Figure 4 summarizes the effects we have discussed so far. The solid line traces the fraction of allowed initial conditions, sampled according to the assumptions stated in §3.2, that generate 2-planet scenarios which survive for  $t = 10^6$  yr. Nearly co-planar scenarios, whether co-rotating or counter-rotating, betray little or no sign of instability. Those co-rotating systems at  $\Theta_0 \leq 30^\circ$  that are unstable all have  $\sin i \lesssim 0.3$  for either planet C or D. Our estimated fraction of surviving systems is an upper limit because we have neglected thus far the presence of planet B and because there may be instabilities that require  $t > 10^6$  yr to develop. We expect the survival fraction to be most severely overestimated at  $40^\circ \lesssim \Theta_0 \lesssim 140^\circ$ , inclinations for which  $e_C$  and  $\Theta$  undergo large, coupled variations; values of  $e_C \gtrsim 0.93$  can cause planets C and B to undergo destabilizing close encounters.

Open circles in Figure 4 mark  $P_{time}$  ( $|\Delta\omega| \leq 10^\circ$ ), the fraction of time a surviving sampled scenario spends with  $|\Delta\omega| \leq 10^\circ$ . For  $\Theta_0 \lesssim 20^\circ$ , the two planets are locked in a secular apsidal resonance for which  $P_{time}$  can be as high as 0.3–0.4. For  $\Theta_0 \gtrsim 40^\circ$ ,  $\Delta\omega$  typically circulates so that  $P_{time} = 20^\circ/360^\circ = 0.055$ . The specific distribution of values of  $P_{time}$  at a given  $\Theta_0$  depends upon our assumed sampling function for  $\chi_0$ , the initial argument of pericenter (see §3.2). For given  $\Theta_0$ , lower values of  $P_{time}$  typically correspond to lower values of  $\sin i_C$  and  $\sin i_D$  (see §2.2). Maximum values of  $P_{time} \approx 0.39$  are attained at  $\Theta_0 = 10$ – $20^\circ$ .

Surprisingly, a secular apsidal resonance exists at  $\Theta_0 \approx 70$ – $80^\circ$  for which  $P_{time} \approx 0.2$ . The evolution within this “anomalous” resonance at high  $\Theta_0$  is characterized by rapid oscillations in  $\Delta\omega$ ,  $e_C$ ,  $e_D$ , and  $\Theta$  at twice the nodal precession frequency,  $\sim 2|\dot{\Omega}|$ . The eccentricity of planet C is periodically driven by secular perturbations to values as high as 0.95. Given such large values of  $e_C$ , we suspect that including the presence of planet B may render these anomalously resonant configurations unstable. We confirm this suspicion in the next section.

Fraction of Surviving 2-Planet Scenarios



#### 4. Planets B, C, and D

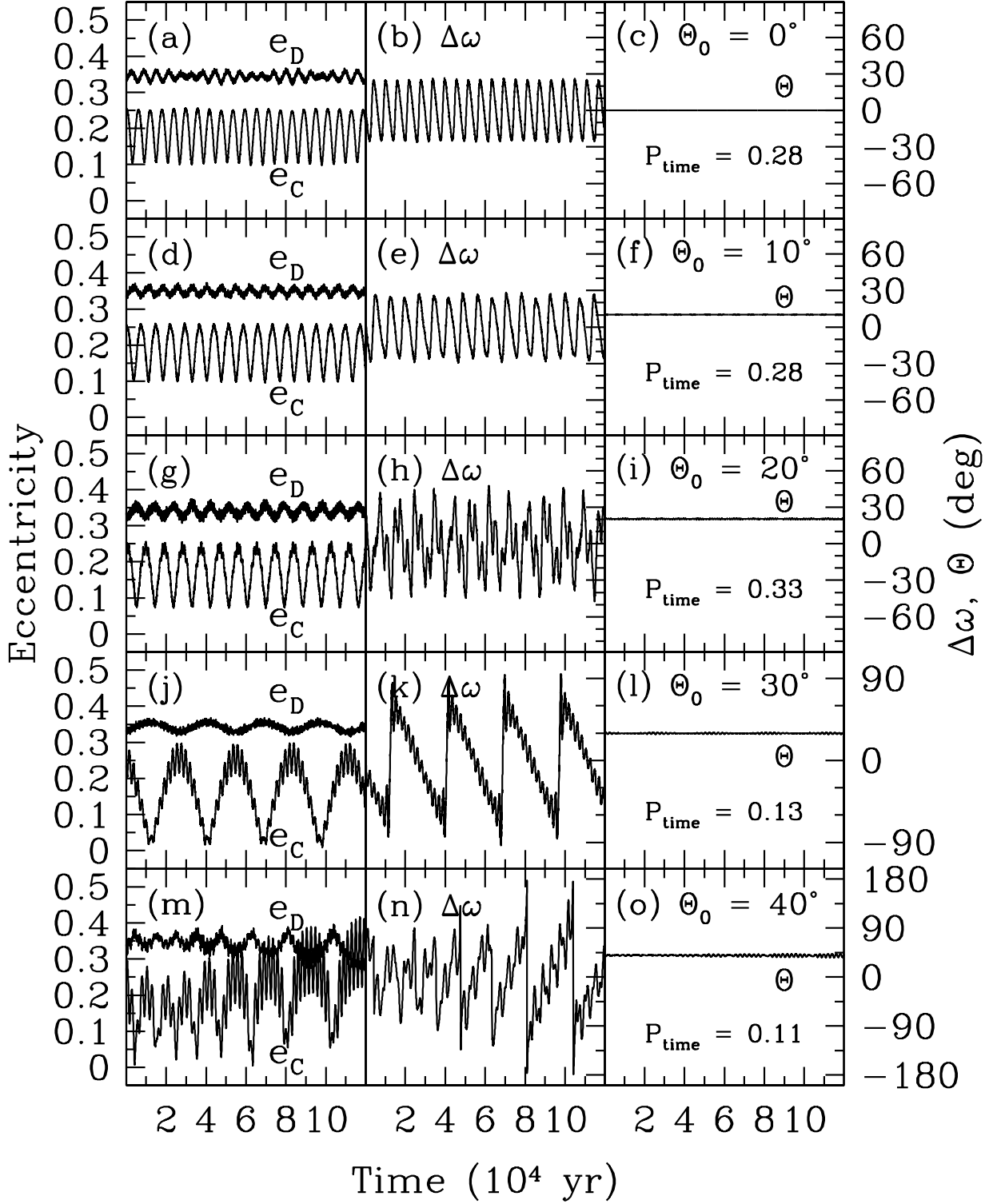
In the previous section we found those allowed initial conditions involving only planets C and D that result in libration of  $\Delta\omega$ . Here we add the effects of the third, innermost planet B to these resonant configurations, in the expectation that some of the resultant 3-planet systems will not be stable.

The inclusion of planet B introduces a number of complications. First, three extra dimensions are added to the space of allowed initial orientations:  $\Theta_{B0}$ ,  $\Omega_{B0}$ , and  $\chi_{B0}$ —respectively, the initial inclination, initial longitude of ascending node, and initial argument of pericenter of the orbit of planet B upon the reference orbit of planet C (see §3.1). Since the present-day orbit of planet B appears nearly indistinguishable from a circle (see Table 1), we can take  $e_{B0} = 0$  to eliminate  $\chi_{B0}$  as a dimension. Even with this simplifying assumption, the volume of additional parameter space to be surveyed is, in principle, substantial. Computational needs are further exacerbated by the fact that the orbital period of planet B is 4.6 days; this necessitates a commensurately short computational timestep. Finally, planet B is sufficiently close to its parent star that contributions to the planet’s apsidal and nodal precession rates due to stellar oblateness and general relativistic effects are not negligible compared to contributions from the outer two planets.

We proceed with a simplified program. First, we select a set of allowed initial orientations for planets C and D which give rise to apsidally resonant, 2-planet scenarios. At each  $\Theta_0$ , we select the one 2-planet scenario that generates the maximum value of  $P_{time}$  ( $|\Delta\omega| \leq 10^\circ$ ), provided this quantity exceeds 0.15. As evident in Figure 4, the values of  $\Theta_0$  which give rise to such scenarios are  $0^\circ$ ,  $10^\circ$ ,  $20^\circ$ ,  $30^\circ$ ,  $40^\circ$ ,  $70^\circ$ , and  $80^\circ$ . Next, to each member of this set of allowed initial orientations, we add the following orientation angles for planet B:  $\Theta_{B0} = \Theta_0$  and  $\Omega_{B0} = \Omega_0$ . That is, the orbit of planet B is assumed to be initially circular and to lie in the orbit plane of the most massive planet, D. Finally, this augmented set of allowed initial orientations is combined with the remaining measured orbital parameters for all three planets and employed as initial conditions for the SWIFT orbit integrator. We employ a timestep of  $\Delta t = 0.20$  days (5% of the orbital period of planet B) and run each integration for  $t = 10^6$  yr. The effects of stellar oblateness and general relativity are ignored.

While the above procedure does not permit accurate determination of the long-term orbital evolution of planet B, we believe it does capture qualitatively the effect of planet B on the possibility of resonant apsidal lock between planets C and D. We expect the innermost planet to represent primarily a potential means of destroying the apsidal resonance, as the secular growth of  $e_C$  takes planet C disruptively close to planet B.

Scenarios for which all three planets survive the entire duration of the integration are shown in Figure 5. Only scenarios for which  $\Theta_0 = \Theta_{B0} \leq 40^\circ$  qualify. For  $\Theta_0 \geq 70^\circ$ , the eccentricity of planet C is secularly driven to such high values that planet B is perturbed by planet C into a star-crossing orbit in  $t \sim 10^4$  yr. Thus, the “anomalous” resonant configurations at  $\Theta_0 = 70^\circ$ – $80^\circ$  found in §3.3 do not survive the inclusion of planet B. A similar fate probably befalls the scenario



for which  $\Theta_0 = 40^\circ$ ; an orbit integration of duration  $t > 10^6$  yr is required to confirm this suspicion.

For  $\Theta_0 \leq 30^\circ$ , planets C and D can remain in stable resonant lock despite the presence of planet B. Values of  $P_{time}$  ( $|\Delta\omega| \leq 10^\circ$ ) in 3-planet scenarios are somewhat reduced from their respective values in 2-planet scenarios. Values of  $P_{time} \approx 0.28$ – $0.33$  obtain for  $\Theta_0 \leq 20^\circ$ ; at  $\Theta_0 = 30^\circ$ ,  $P_{time}$  drops to 0.13.

## 5. Concluding Remarks

Our results suggest that the outer two planets of Upsilon Andromedae occupy nearly co-planar, co-rotating orbits. The lines of evidence are as follows:

1. Finding the Doppler measured  $\omega$ 's of planets C and D to be nearly equal today is least surprising if the mutual orbit inclination  $\Theta_0 \lesssim 20^\circ$  (Figures 4 and 5). If  $\Theta_0 \lesssim 20^\circ$ , planets C and D can be locked in a secular apsidal resonance for which  $\Delta\omega \equiv \omega_D - \omega_C$  librates about  $0^\circ$  with an amplitude of approximately  $30^\circ$ . The corresponding fraction of time that  $|\Delta\omega| \leq 10^\circ$  is  $P_{time} \approx 0.30$ . These libration parameters are subject to change with future refinements in the values of the present-day fitted eccentricities (Figure 3). By contrast, if  $\Theta_0 \gtrsim 40^\circ$ , typically  $\Delta\omega$  circulates and  $P_{time} \approx 20^\circ/360^\circ = 0.055$  for the relatively few allowed initial conditions that give rise to stable systems.
2. The secular apsidal resonance at  $\Theta_0 \lesssim 20^\circ$  limits variations in eccentricities and thereby affords the system stability (Figure 2). By contrast, if  $40^\circ \lesssim \Theta_0 \lesssim 90^\circ$ , secular driving of the eccentricity of planet C to values approaching unity via the Kozai resonance fosters close approaches and concomitant instability. If  $90^\circ \lesssim \Theta_0 \lesssim 140^\circ$ , the destabilizing effects of coupled  $e_C$  and  $\Theta$  are abetted by synchronized driving of the eccentricities.

We note that co-planar, counter-rotating configurations ( $\Theta \approx 180^\circ$ ) are allowed by argument 2 but disfavored by argument 1. Furthermore, values of  $\sin i_C$ ,  $\sin i_D \gtrsim 0.5$  are favored, not only for reasons of stability, but also because such values are typical of scenarios for which the fraction of time spent with small  $|\Delta\omega|$  is maximized.

The preference for small mutual inclination, based on combining considerations of stability with the likelihood of finding  $|\Delta\omega| \ll 1$  rad, is consistent with the origin of these planets in a flattened, circumstellar disk. How does our expectation that  $\Theta_0 \lesssim 20^\circ$  compare with orbital inclinations in related contexts? The opening angle of hydrostatically flared protoplanetary disks at stellocentric distances of  $\sim 1$  AU is  $\sim 6^\circ$  (see, e.g., Chiang & Goldreich 1997). The planets in the Solar System, with the understandable exception of Pluto (Malhotra 1998), occupy orbits whose mutual inclinations do not exceed about 7 degrees; Mercury aside, mutual inclinations do not exceed about 3 degrees.



By contrast, scenarios for the formation of extrasolar planets that do not involve a primordial disk may predict a substantial population of systems for which  $\Theta \gtrsim 150^\circ$ . For example, Papaloizou & Terquem (2000) have proposed that extrasolar planets might gravitationally fragment from, and scatter out of, a circumstellar spherical cloud of gas. Planetary systems having a wide distribution of initial mutual inclinations from  $\Theta_0 = 0^\circ$  to  $180^\circ$  might be expected to form. From the stability considerations elucidated above, we might expect a bimodal population of systems to survive—those for which  $0^\circ \lesssim \Theta \lesssim 40^\circ$  and those for which  $150^\circ \lesssim \Theta \lesssim 180^\circ$ —intermediate inclinations having been eliminated via the destructive Kozai effect.

The final word on the actual values of orbital inclinations in extrasolar planetary systems will likely come from combining the Doppler radial velocity data with astrometric measurements, or, in rare cases, from transit observations (Mazeh et al. 2000; Queloz et al. 2000). The amplitudes of stellar wobbles induced by planets C and D are  $(0.0943/\sin i_C)$  mas and  $(0.576/\sin i_D)$  mas, respectively (Pourbaix 2001). The star  $\nu$  And is sufficiently bright ( $V = 4.2$ ) that the upcoming astrometric satellite FAME (Full-Sky Astrometric Mapping Explorer) should yield at least a  $2\sigma$  detection of the astrometric signature of planet C.

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Fig. 1.— Defining geometry for the Doppler-measured argument of pericenter  $\omega$ , the angle between pericenter and the node of the planetary orbit on the plane of the sky. The sky plane selected is that which passes through the system barycenter. The vector  $\hat{\mathbf{n}}$  lies in the planes of the orbit and of the sky. The vector  $\hat{\mathbf{e}}$  lies in the orbit plane. The node selected is the one for which the planet travels towards the observer. The argument  $\omega$  advances along the orbit plane in the direction of increasing true anomaly.

Fig. 2.— Sampling of 2-planet scenarios in which the orbits of planets C and D are initially inclined by the value of  $\Theta_0$  indicated. Each horizontal row of panels displays the evolution of  $e_C$ ,  $e_D$ ,  $\Delta\omega = \omega_D - \omega_C$ , and  $\Theta = \arccos(\hat{\mathbf{I}}_C \cdot \hat{\mathbf{I}}_D)$  for a given scenario. For panels (a)–(c), the evolution is computed using the classical, analytic solution of Laplace-Lagrange for  $\Theta = 0^\circ$ . For the remaining panels, the evolution is computed by numerical integration. Libration of  $\Delta\omega$  is a typical outcome for  $\Theta_0 \lesssim 20^\circ$ , while circulation of  $\Delta\omega$  and large secular variations of  $e_C$  and  $\Theta$  via the Kozai mechanism are possible outcomes for  $40^\circ \lesssim \Theta_0 \lesssim 140^\circ$ . For  $\Theta_0 \gtrsim 150^\circ$ ,  $\Delta\omega$  circulates while  $e_C$  and  $e_D$  vary modestly and synchronously. Note in panel (f) how  $\Delta\omega$  is modulated at a number of frequencies, reflecting its dependence on the 4 precessing vectors  $(\hat{\mathbf{I}}_C)_{C,D}$  and  $(\hat{\mathbf{e}})_{C,D}$ . For panels (a)–(f),  $\sin i = 1$  and timescales scale as  $\sin i$ . For panels (g)–(i),  $\sin i_C = 0.99$  and  $\sin i_D = 0.93$ . For panels (j)–(l),  $\sin i_C = 1$  and  $\sin i_D = 0.73$ . For panels (m)–(o),  $\sin i_C = 0.78$  and  $\sin i_D = 0.87$ .

Fig. 3.— Contours of constant libration amplitude in degrees as a function of possible present-day fitted eccentricities, for  $\Theta = 0^\circ$  and  $\sin i = 1$ . The cross indicates the best-fit eccentricities from Marcy & Fischer (2001), while the inner and outer dotted ellipses enclose the  $1\sigma$  and  $2\sigma$  confidence regions for the eccentricities, respectively. Panel (a) is calculated using Laplace-Lagrange theory while panel (b) plots the maximum value of  $\Delta\omega$  over  $t = 10^5$  yr in numerical integrations using SWIFT.

Fig. 4.— *Open circles, left-hand ordinate*: Fraction of time that surviving, sampled, 2-planet scenarios spend with  $|\Delta\omega| \leq 10^\circ$ . The observation today that  $|\Delta\omega| \leq 10^\circ$  is least surprising if the mutual inclination  $\Theta$  between the orbit planes of planets C and D is less than or equal to  $20^\circ$ . At these modest inclinations, the two planets inhabit a secular apsidal resonance in which  $\Delta\omega$  librates about  $0^\circ$  with an amplitude of  $\sim 25^\circ$ . The cluster of points at  $\Theta = 70^\circ$ – $80^\circ$  and relatively large  $P_{\text{time}} \approx 0.20$  reflects the existence of an “anomalous” apsidal 2-planet resonance. In §4, we discuss how the inclusion of the third, innermost planet B disrupts the anomalous resonance. *Solid line, right-hand ordinate*: Fraction of 2-planet scenarios that survive the  $10^6$  yr duration of the integration. Nearly co-planar orbits, whether co-rotating or counter-rotating, betray little sign of instability. Scenarios at intermediate inclinations tend to be unstable because the Kozai effect at  $\Theta \gtrsim 40^\circ$ , abetted by synchronous driving of eccentricities at  $\Theta \gtrsim 90^\circ$ , drives the eccentricity of planet C to such high values that planet C collides with the star.

Fig. 5.— Three-planet scenarios in which planets B and D initially occupy co-planar orbits while planet C initially occupies an orbit inclined by the value of  $\Theta_0$  indicated. Each horizontal row

of panels displays the evolution of  $e_C$ ,  $e_D$ ,  $\Delta\omega = \omega_D - \omega_C$ , and  $\Theta = \arccos(\hat{\mathbf{l}}_C \cdot \hat{\mathbf{l}}_D)$  for a given scenario. Scenarios for which  $\Theta_0 \leq 20^\circ$  preserve apsidal libration of  $\Delta\omega$  despite the presence of planet B and spend the maximum fraction of time with  $|\Delta\omega| \leq 10^\circ$ :  $P_{time} \approx 0.28$ – $0.33$ . At  $\Theta_0 = 30^\circ$ , the amplitude of apsidal libration is at its largest, approximately  $90^\circ$ . At  $\Theta_0 \geq 40^\circ$ ,  $\Delta\omega$  eventually circulates. For panels (a)–(c),  $\sin i_B = \sin i_C = \sin i_D = 0.89$ . For panels (d)–(f),  $\sin i_B = \sin i_D = 0.99$  and  $\sin i_C = 0.95$ . For panels (g)–(i),  $\sin i_B = \sin i_D = 0.79$  and  $\sin i_C = 0.55$ . For panels (j)–(l),  $\sin i_B = \sin i_D = 0.99$  and  $\sin i_C = 0.97$ . For panels (m)–(o),  $\sin i_B = \sin i_D = 0.90$  and  $\sin i_C = 0.44$ .

Table 1. Fitted Orbital Parameters of Upsilon Andromedae (Marcy & Fischer 2001)<sup>ab</sup>

Planet	$P$ (days)	$T_{peri}$ (JD)	$e$	$\omega$	$K$ (m/s)	$m \sin i$ ( $M_J$ )	$a$ (AU)
B	4.61706 ( $7 \times 10^{-5}$ )	2450001.6 (NA) <sup>c</sup>	0.015 (0.009)	32.1 (NA) <sup>c</sup>	71.0 (0.7)	0.69	0.059
C	241.14 (0.22)	2450160.1 (1.9)	0.254 (0.015)	-109.0 (2.9)	55.6 (0.9)	1.96	0.828
D	1309.13 (5.13)	2450044.0 (11.3)	0.342 (0.015)	-104.2 (3.8)	66.2 (1.2)	3.98	2.557

<sup>a</sup>Based on fitting  $N = 189$  radial velocity points, with rms residuals of 11.5 m/s.

<sup>b</sup>The orbital period is  $P$ , the Julian date of pericenter passage is  $T_{peri}$ , the eccentricity is  $e$ , the planetary mass is  $m$  measured in Jupiter masses  $M_J$ , and the semi-major axis is  $a$ . Remaining variables are defined in the text. Uncertainties ( $1\sigma$ ) in fitted values are enclosed in parentheses.

<sup>c</sup>Uncertainties in  $\omega$  and  $T_{peri}$  have little meaning for the nearly circular orbit of planet B.